

MAXWELL MEETS KORN: A NEW COERCIVE INEQUALITY FOR TENSOR FIELDS IN $\mathbb{R}^{N \times N}$ WITH SQUARE-INTEGRABLE EXTERIOR DERIVATIVE

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January 25, 2013

Abstract

For a bounded domain $\Omega \subset \mathbb{R}^N$ with connected Lipschitz boundary we prove the existence of some $c > 0$, such that

$$c \|P\|_{L^2(\Omega, \mathbb{R}^{N \times N})} \leq \|\text{sym } P\|_{L^2(\Omega, \mathbb{R}^{N \times N})} + \|\text{Curl } P\|_{L^2(\Omega, \mathbb{R}^{N \times (N-1)N/2})}$$

holds for all square-integrable tensor fields $P : \Omega \rightarrow \mathbb{R}^{N \times N}$, having square-integrable generalized ‘rotation’ $\text{Curl } P : \Omega \rightarrow \mathbb{R}^{N \times (N-1)N/2}$ and vanishing tangential trace on $\partial\Omega$, where both operations are to be understood row-wise. Here, in each row the operator curl is the vector analytical reincarnation of the exterior derivative d in \mathbb{R}^N . For compatible tensor fields P , i.e., $P = \nabla v$, the latter estimate reduces to a non-standard variant of Korn’s first inequality in \mathbb{R}^N , namely

$$c \|\nabla v\|_{L^2(\Omega, \mathbb{R}^{N \times N})} \leq \|\text{sym } \nabla v\|_{L^2(\Omega, \mathbb{R}^{N \times N})}$$

for all vector fields $v \in H^1(\Omega, \mathbb{R}^N)$, for which ∇v_n , $n = 1, \dots, N$, are normal at $\partial\Omega$.

Key Words Korn’s inequality, theory of Maxwell equations in \mathbb{R}^N , Helmholtz decomposition, Poincaré/Friedrichs type estimates

1 Introduction and Preliminaries

We extend the results from [12], which have been announced in [13], to the N -dimensional case following in close lines the arguments presented there. Let $N \in \mathbb{N}$ and Ω be a bounded domain in \mathbb{R}^N with connected Lipschitz boundary $\Gamma := \partial\Omega$. We prove a Korn-type inequality in $\mathring{H}(\text{Curl}; \Omega)$ for eventually non-symmetric tensor fields P mapping Ω to $\mathbb{R}^{N \times N}$. More precisely, there exists a positive constant c , such that

$$c \|P\|_{L^2(\Omega)} \leq \|\text{sym } P\|_{L^2(\Omega)} + \|\text{Curl } P\|_{L^2(\Omega)}$$

holds for all tensor fields $P \in \mathring{\mathbf{H}}(\text{Curl}; \Omega)$, where P belongs to $\mathring{\mathbf{H}}(\text{Curl}; \Omega)$, if $P \in \mathbf{H}(\text{Curl}; \Omega)$ has vanishing tangential trace on Γ . Thereby, the generalized Curl and tangential trace are defined as row-wise operations. For compatible tensor fields $P = \nabla v$ with vector fields $v \in \mathbf{H}^1(\Omega)$, for which ∇v_n , $n = 1, \dots, N$, are normal at $\partial\Omega$, the latter estimate reduces to a non-standard variant of the well known Korn's first inequality in \mathbb{R}^N

$$c \|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}.$$

Our proof relies on three essential tools, namely

1. Maxwell estimate (Poincaré-type estimate),
2. Helmholtz' decomposition,
3. Korn's first inequality.

In [12] we already pointed out the importance of the Maxwell estimate and the related question of the Maxwell compactness property*. Here, we mention the papers [2, 6, 15, 16, 17, 18, 20]. Results for the Helmholtz decomposition can be found in [3, 14, 15, 17, 20, 19, 7, 8, 9]. Nowadays, differential forms find prominent applications in numerical methods like Finite Element Exterior Calculus [1, 4] or Discrete Exterior Calculus [5].

1.1 Differential Forms

We may look at Ω as a smooth Riemannian manifold of dimension N with compact closure and connected Lipschitz continuous boundary Γ . The alternating differential forms of rank $q \in \{0, \dots, N\}$ on Ω , briefly q -forms, with square-integrable coefficients will be denoted by $\mathbf{L}^{2,q}(\Omega)$. The exterior derivative d and the co-derivative $\delta = \pm * d *$ ($*$: Hodge's star operator) are formally skew-adjoint to each other, i.e.,

$$\forall E \in \mathring{\mathbf{C}}^{\infty,q}(\Omega) \quad H \in \mathring{\mathbf{C}}^{\infty,q+1}(\Omega) \quad \langle dE, H \rangle_{\mathbf{L}^{2,q+1}(\Omega)} = - \langle E, \delta H \rangle_{\mathbf{L}^{2,q}(\Omega)},$$

where the $\mathbf{L}^{2,q}(\Omega)$ -scalar product is given by

$$\forall E, H \in \mathbf{L}^{2,q}(\Omega) \quad \langle E, H \rangle_{\mathbf{L}^{2,q}(\Omega)} := \int_{\Omega} E \wedge *H.$$

Here $\mathring{\mathbf{C}}^{\infty,q}(\Omega)$ denotes the space of compactly supported and smooth q -forms on Ω . Using this duality, we can define weak versions of d and δ . The corresponding standard Sobolev spaces are denoted by

$$\begin{aligned} D^q(\Omega) &:= \{E \in \mathbf{L}^{2,q}(\Omega) : dE \in \mathbf{L}^{2,q+1}(\Omega)\}, \\ \Delta^q(\Omega) &:= \{H \in \mathbf{L}^{2,q}(\Omega) : \delta H \in \mathbf{L}^{2,q-1}(\Omega)\}. \end{aligned}$$

*By 'Maxwell estimate' and 'Maxwell compactness property' we mean the estimates and compact embedding results used in the theory of Maxwell's equations.

The homogeneous tangential boundary condition $\tau_\Gamma E = 0$, where τ_Γ denotes the tangential trace, is generalized in the space

$$\mathring{D}^q(\Omega) := \overline{\mathring{C}^{\infty,q}(\Omega)},$$

where the closure is taken in $D^q(\Omega)$. In classical terms, we have for smooth q -forms $\tau_\Gamma = \iota^*$ with the canonical embedding $\iota : \Gamma \hookrightarrow \overline{\Omega}$. An index 0 at the lower right position indicates vanishing derivatives, i.e.,

$$\mathring{D}_0^q(\Omega) = \{E \in \mathring{D}^q(\Omega) : dE = 0\}, \quad \Delta_0^q(\Omega) = \{H \in \Delta^q(\Omega) : \delta H = 0\}.$$

By definition and density, we have

$$\Delta_0^q(\Omega) := (d\mathring{D}^{q-1}(\Omega))^\perp, \quad \Delta_0^q(\Omega)^\perp := \overline{d\mathring{D}^{q-1}(\Omega)},$$

where \perp denotes the orthogonal complement with respect to the $L^{2,q}(\Omega)$ -scalar product and the closure is taken in $L^{2,q}(\Omega)$. Hence, we obtain the $L^{2,q}(\Omega)$ -orthogonal decomposition, usually called Hodge-Helmholtz decomposition,

$$L^{2,q}(\Omega) = \overline{d\mathring{D}^{q-1}(\Omega)} \oplus \Delta_0^q(\Omega), \quad (1.1)$$

where \oplus denotes the orthogonal sum with respect to the $L^{2,q}(\Omega)$ -scalar product. In [20, 16] the following crucial tool has been proved:

Lemma 1 (Maxwell Compactness Property) *For all q the embeddings*

$$\mathring{D}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow L^{2,q}(\Omega)$$

are compact.

As a first immediate consequence, the spaces of so called ‘harmonic Dirichlet forms’

$$\mathcal{H}^q(\Omega) := \mathring{D}_0^q(\Omega) \cap \Delta_0^q(\Omega)$$

are finite dimensional. In classical terms, a q -form E belongs to $\mathcal{H}^q(\Omega)$, if

$$dE = 0, \quad \delta E = 0, \quad \iota^* E = 0.$$

The dimension of $\mathcal{H}^q(\Omega)$ equals the $(N - q)$ th Betti number of Ω . Since we assume the boundary Γ to be connected, the $(N - 1)$ th Betti number of Ω vanishes and therefore there are no Dirichlet forms of rank 1 besides zero, i.e.,

$$\mathcal{H}^1(\Omega) = \{0\}. \quad (1.2)$$

This condition on the domain Ω resp. its boundary Γ is satisfied e.g. for a ball or a torus.

By a usual indirect argument, we achieve another immediate consequence:

Lemma 2 (Poincaré Estimate for Differential Forms) *For all q there exist positive constants $c_{p,q}$, such that for all $E \in \mathring{D}^q(\Omega) \cap \Delta^q(\Omega) \cap \mathcal{H}^q(\Omega)^\perp$*

$$\|E\|_{L^{2,q}(\Omega)} \leq c_{p,q} \left(\|dE\|_{L^{2,q+1}(\Omega)}^2 + \|\delta E\|_{L^{2,q-1}(\Omega)}^2 \right)^{1/2}.$$

Since

$$d\mathring{D}^{q-1}(\Omega) \subset \mathring{D}_0^q(\Omega)$$

(note that $dd=0$ and $\delta\delta=0$ hold even in the weak sense) we get by (1.1)

$$d\mathring{D}^{q-1}(\Omega) = d(\mathring{D}^{q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega)) = d(\mathring{D}^{q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega) \cap \mathcal{H}^{q-1}(\Omega)^\perp).$$

Now, Lemma 2 shows that $d\mathring{D}^{q-1}(\Omega)$ is already closed. Hence, we obtain a refinement of (1.1)

Lemma 3 (Hodge-Helmholtz Decomposition for Differential Forms) *The decomposition*

$$L^{2,q}(\Omega) = d\mathring{D}^{q-1}(\Omega) \oplus \Delta_0^q(\Omega)$$

holds.

1.2 Functions and Vector Fields

Let us turn to the special case $q = 1$. In this case, we choose (e.g.) the identity as single global chart for Ω and use the canonical identification isomorphism for 1-forms (i.e., Riesz' representation theorem) with vector fields $dx_n \cong e^n$, namely

$$\sum_{n=1}^N v_n(x) dx_n \cong v(x) = \begin{bmatrix} v_1(x) \\ \vdots \\ v_N(x) \end{bmatrix}, \quad x \in \Omega.$$

0-forms will be isomorphically identified with functions on Ω . Then, $d \cong \text{grad} = \nabla$ for 0-forms (functions) and $\delta \cong \text{div} = \nabla \cdot$ for 1-forms (vector fields). Hence, the well known first order differential operators from vector analysis occur. Moreover, on 1-forms we define a new operator $\text{curl} : \cong \mathfrak{d}$, which turns into the usual curl if $N = 3$ or $N = 2$. $L^{2,q}(\Omega)$ equals the usual Lebesgue spaces of square integrable functions or vector fields on Ω with values in \mathbb{R}^n , $n := n_{N,q} := \binom{N}{q}$, which will be denoted by $L^2(\Omega) := L^2(\Omega, \mathbb{R}^n)$. $D^0(\Omega)$ and $\Delta^1(\Omega)$ are identified with the standard Sobolev spaces

$$H(\text{grad}; \Omega) := \{u \in L^2(\Omega, \mathbb{R}) : \text{grad } u \in L^2(\Omega, \mathbb{R}^N)\} = H^1(\Omega),$$

$$H(\text{div}; \Omega) := \{v \in L^2(\Omega, \mathbb{R}^N) : \text{div } v \in L^2(\Omega, \mathbb{R})\},$$

respectively. Moreover, we may now identify $\mathbf{D}^1(\Omega)$ with

$$\mathbf{H}(\text{curl}; \Omega) := \{v \in \mathbf{L}^2(\Omega, \mathbb{R}^N) : \text{curl } v \in \mathbf{L}^2(\Omega, \mathbb{R}^{(N-1)N/2})\},$$

which is the well known $\mathbf{H}(\text{curl}; \Omega)$ for $N = 2, 3$. E.g., for $N = 4$ we have

$$\text{curl } v = \begin{bmatrix} \partial_1 v_2 - \partial_2 v_1 \\ \partial_1 v_3 - \partial_3 v_1 \\ \partial_1 v_4 - \partial_4 v_1 \\ \partial_2 v_3 - \partial_3 v_2 \\ \partial_2 v_4 - \partial_4 v_2 \\ \partial_3 v_4 - \partial_4 v_3 \end{bmatrix} \in \mathbb{R}^6$$

and for $N = 5$ we get $\text{curl } v \in \mathbb{R}^{10}$. In general, the entries of the $(N-1)N/2$ -vector $\text{curl } v$ consist of all possible combinations of

$$\partial_n v_m - \partial_m v_n, \quad 1 \leq n < m \leq N.$$

Similarly, we obtain the closed subspaces

$$\mathring{\mathbf{H}}(\text{grad}; \Omega) = \mathring{\mathbf{H}}^1(\Omega), \quad \mathring{\mathbf{H}}(\text{curl}; \Omega)$$

as reincarnations of $\mathring{\mathbf{D}}^0(\Omega)$ and $\mathring{\mathbf{D}}^1(\Omega)$, respectively. We note

$$\mathring{\mathbf{H}}(\text{grad}; \Omega) = \overline{\mathring{\mathbf{C}}^\infty(\Omega)}, \quad \mathring{\mathbf{H}}(\text{curl}; \Omega) = \overline{\mathring{\mathbf{C}}^\infty(\Omega)},$$

where the closures are taken in the respective graph norms, and that in these Sobolev spaces the classical homogeneous scalar and tangential (compare to $N = 3$) boundary conditions

$$u|_\Gamma = 0, \quad \nu \times v|_\Gamma = 0$$

are generalized. Here, ν denotes the outward unit normal for Γ . Furthermore, we have the spaces of irrotational or solenoidal vector fields

$$\mathbf{H}(\text{curl}_0; \Omega) = \{v \in \mathbf{H}(\text{curl}; \Omega) : \text{curl } v = 0\},$$

$$\mathring{\mathbf{H}}(\text{curl}_0; \Omega) = \{v \in \mathring{\mathbf{H}}(\text{curl}; \Omega) : \text{curl } v = 0\},$$

$$\mathbf{H}(\text{div}_0; \Omega) = \{v \in \mathbf{H}(\text{div}; \Omega) : \text{div } v = 0\}.$$

Again, all these spaces are Hilbert spaces. Now, we have two compact embeddings

$$\mathring{\mathbf{H}}(\text{grad}; \Omega) \hookrightarrow \mathbf{L}^2(\Omega), \quad \mathring{\mathbf{H}}(\text{curl}; \Omega) \cap \mathbf{H}(\text{div}; \Omega) \hookrightarrow \mathbf{L}^2(\Omega),$$

i.e., Rellich's selection theorem and the Maxwell compactness property. Moreover, the following Poincaré and Maxwell estimates hold:

Corollary 4 (Poincaré Estimate for Functions) *Let $c_p := c_{p,0}$. Then, for all functions $u \in \mathring{H}(\text{grad}; \Omega)$*

$$\|u\|_{L^2(\Omega)} \leq c_p \|\text{grad } u\|_{L^2(\Omega)}.$$

Corollary 5 (Maxwell Estimate for Vector Fields) *Let $c_m := c_{p,1}$. Then, for all vector fields $v \in \mathring{H}(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$*

$$\|v\|_{L^2(\Omega)} \leq c_m \left(\|\text{curl } v\|_{L^2(\Omega)}^2 + \|\text{div } v\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

We note that generally $\mathcal{H}^0(\Omega) = \{0\}$ and by (1.2) also $\mathcal{H}^1(\Omega) = \{0\}$. The appropriate Helmholtz decomposition for our needs is

Corollary 6 (Helmholtz Decomposition for Vector Fields)

$$L^2(\Omega) = \text{grad } \mathring{H}(\text{grad}; \Omega) \oplus H(\text{div}_0; \Omega)$$

1.3 Tensor Fields

We extend our calculus to $(N \times N)$ -tensor (matrix) fields. For vector fields v with components in $H(\text{grad}; \Omega)$ and tensor fields P with rows in $H(\text{curl}; \Omega)$ resp. $H(\text{div}; \Omega)$, i.e.,

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad v_n \in H(\text{grad}; \Omega), \quad P = \begin{bmatrix} P_1^T \\ \vdots \\ P_N^T \end{bmatrix}, \quad P_n \in H(\text{curl}; \Omega) \text{ resp. } H(\text{div}; \Omega)$$

for $n = 1, \dots, N$, we define

$$\text{Grad } v := \begin{bmatrix} \text{grad}^T v_1 \\ \vdots \\ \text{grad}^T v_N \end{bmatrix} = J_v = \nabla v, \quad \text{Curl } P := \begin{bmatrix} \text{curl}^T P_1 \\ \vdots \\ \text{curl}^T P_N \end{bmatrix}, \quad \text{Div } P := \begin{bmatrix} \text{div } P_1 \\ \vdots \\ \text{div } P_N \end{bmatrix},$$

where J_v denotes the Jacobian of v and T the transpose. We note that v and $\text{Div } P$ are N -vector fields, P and $\text{Grad } v$ are $(N \times N)$ -tensor fields, whereas $\text{Curl } P$ is a $(N \times (N-1)N/2)$ -tensor field which may also be viewed as a totally anti-symmetric third order tensor field with entries

$$(\text{Curl } P)_{ijk} = \partial_j P_{ik} - \partial_k P_{ij}.$$

The corresponding Sobolev spaces will be denoted by

$$\begin{array}{llll} H(\text{Grad}; \Omega), & \mathring{H}(\text{Grad}; \Omega), & H(\text{Div}; \Omega), & H(\text{Div}_0; \Omega), \\ H(\text{Curl}; \Omega), & \mathring{H}(\text{Curl}; \Omega), & H(\text{Curl}_0; \Omega), & \mathring{H}(\text{Curl}_0; \Omega). \end{array}$$

There are three crucial tools to prove our estimate. First, we have obvious consequences from Corollaries 4, 5 and 6:

Corollary 7 (Poincaré Estimate for Vector Fields) *For all $v \in \mathring{\mathbf{H}}(\text{Grad}; \Omega)$*

$$\|v\|_{\mathbf{L}^2(\Omega)} \leq c_p \|\text{Grad } v\|_{\mathbf{L}^2(\Omega)}.$$

Corollary 8 (Maxwell Estimate for Tensor Fields) *The estimate*

$$\|P\|_{\mathbf{L}^2(\Omega)} \leq c_m \left(\|\text{Curl } P\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{Div } P\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}$$

holds for all tensor fields $P \in \mathring{\mathbf{H}}(\text{Curl}; \Omega) \cap \mathbf{H}(\text{Div}; \Omega)$.

Corollary 9 (Helmholtz Decomposition for Tensor Fields)

$$\mathbf{L}^2(\Omega) = \text{Grad } \mathring{\mathbf{H}}(\text{Grad}; \Omega) \oplus \mathbf{H}(\text{Div}_0; \Omega)$$

The last important tool is Korn's first inequality.

Lemma 10 (Korn's First Inequality) *For all vector fields $v \in \mathring{\mathbf{H}}(\text{Grad}; \Omega)$*

$$\|\text{Grad } v\|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2} \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)}.$$

Here, we introduce the symmetric and skew-symmetric parts

$$\text{sym } P := \frac{1}{2}(P + P^T), \quad \text{skew } P := \frac{1}{2}(P - P^T)$$

of a $(N \times N)$ -tensor $P = \text{sym } P + \text{skew } P$.

Remark 11 *We note that the proof including the value of the constant is simple. By density we may assume $v \in \mathring{\mathbf{C}}^\infty(\Omega)$. Twofold partial integration yields*

$$\langle \partial_n v_m, \partial_m v_n \rangle_{\mathbf{L}^2(\Omega)} = \langle \partial_m v_m, \partial_n v_n \rangle_{\mathbf{L}^2(\Omega)}$$

and hence

$$\begin{aligned} 2 \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)}^2 &= \frac{1}{2} \sum_{n,m=1}^N \|\partial_n v_m + \partial_m v_n\|_{\mathbf{L}^2(\Omega)}^2 \\ &= \sum_{n,m=1}^N \left(\|\partial_n v_m\|_{\mathbf{L}^2(\Omega)}^2 + \langle \partial_n v_m, \partial_m v_n \rangle_{\mathbf{L}^2(\Omega)} \right) \\ &= \|\text{Grad } v\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{div } v\|_{\mathbf{L}^2(\Omega)}^2 \geq \|\text{Grad } v\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

More on Korn's first inequality can be found, e.g., in [10].

2 Results

For tensor fields $P \in \mathbf{H}(\text{Curl}; \Omega)$ we define the semi-norm

$$\|P\| := \left(\|\text{sym } P\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{Curl } P\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}.$$

The main step is to prove the following

Lemma 12 *Let $\hat{c} := \max\{2, \sqrt{5}c_m\}$. Then, for all $P \in \mathring{\mathbf{H}}(\text{Curl}; \Omega)$*

$$\|P\|_{\mathbf{L}^2(\Omega)} \leq \hat{c} \|P\|.$$

Proof Let $P \in \mathring{\mathbf{H}}(\text{Curl}; \Omega)$. According to Corollary 9 we orthogonally decompose

$$P = \text{Grad } v + S \in \text{Grad } \mathring{\mathbf{H}}(\text{Grad}; \Omega) \oplus \mathbf{H}(\text{Div}_0; \Omega).$$

Then, $\text{Curl } P = \text{Curl } S$ and we observe $S \in \mathring{\mathbf{H}}(\text{Curl}; \Omega) \cap \mathbf{H}(\text{Div}_0; \Omega)$ since

$$\text{Grad } \mathring{\mathbf{H}}(\text{Grad}; \Omega) \subset \mathring{\mathbf{H}}(\text{Curl}_0; \Omega). \quad (2.1)$$

By Corollary 8, we have

$$\|S\|_{\mathbf{L}^2(\Omega)} \leq c_m \|\text{Curl } P\|_{\mathbf{L}^2(\Omega)}. \quad (2.2)$$

Then, by Lemma 10 and (2.2) we obtain

$$\begin{aligned} \|P\|_{\mathbf{L}^2(\Omega)}^2 &= \|\text{Grad } v\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \\ &\leq 2 \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \leq 4 \|\text{sym } P\|_{\mathbf{L}^2(\Omega)}^2 + 5 \|S\|_{\mathbf{L}^2(\Omega)}^2, \end{aligned}$$

which completes the proof. \square

The immediate consequence is our main result

Theorem 13 *On $\mathring{\mathbf{H}}(\text{Curl}; \Omega)$ the norms $\|\cdot\|_{\mathbf{H}(\text{Curl}; \Omega)}$ and $\|\cdot\|$ are equivalent. In particular, $\|\cdot\|$ is a norm on $\mathring{\mathbf{H}}(\text{Curl}; \Omega)$ and there exists a positive constant c , such that*

$$c \|P\|_{\mathbf{H}(\text{Curl}; \Omega)}^2 \leq \|P\|^2 = \|\text{sym } P\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{Curl } P\|_{\mathbf{L}^2(\Omega)}^2$$

holds for all $P \in \mathring{\mathbf{H}}(\text{Curl}; \Omega)$.

Remark 14 *For a skew-symmetric tensor field $P : \Omega \rightarrow \mathfrak{so}(N)$ our estimate reduces to a Poincaré inequality in disguise, since $\text{Curl } P$ controls all partial derivatives of P (compare to [11]) and the homogeneous tangential boundary condition for P is implied by $P|_{\Gamma} = 0$.*

Setting $P := \text{Grad } v$ we obtain

Remark 15 (Korn's First Inequality: Tangential-Variant) *For all $v \in \mathring{H}(\text{Grad}; \Omega)$*

$$\|\text{Grad } v\|_{L^2(\Omega)} \leq \hat{c} \|\text{sym Grad } v\|_{L^2(\Omega)} \quad (2.3)$$

holds by Lemma 12 and (2.1). This is just Korn's first inequality from Lemma 10 with a larger constant \hat{c} . Since Γ is connected, i.e., $\mathcal{H}^1(\Omega) = \{0\}$, we even have

$$\text{Grad } \mathring{H}(\text{Grad}; \Omega) = \mathring{H}(\text{Curl}_0; \Omega).$$

Thus, (2.3) holds for all $v \in H(\text{Grad}; \Omega)$ with $\text{Grad } v \in \mathring{H}(\text{Curl}_0; \Omega)$, i.e., with $\text{Grad } v_n$, $n = 1, \dots, N$, normal at Γ , which then extends Lemma 10 through the (apparently) weaker boundary condition.

The elementary arguments above apply certainly to much more general situations, e.g., to not necessarily connected boundaries Γ and to tangential boundary conditions which are imposed only on parts of Γ . These discussions are left to forthcoming papers.

Acknowledgements We thank the referee for pointing out a missing assumption in a preliminary version of the paper.

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